AEROSOL STATISTICS LOGNORMAL DISTRIBUTIONS AND $dN/dlogD_p$

APPLICATION NOTE PR-001

Lognormal Distributions

Standard statistics based on normal distributions are frequently not suitable for most airborne particle (aerosol) size distributions. Generally, lognormal distributions tend to be the best fit for single source aerosols. There is no real theoretical reason as to why aerosol size distributions are log normal, it's merely empirically the best fit. When statistical analysis is applied to aerosol size distributions, it is routinely based on lognormal distributions. Lognormal statistics are not widely used, and so this slightly skewed point of view from standard statistics can be quite confusing to those who have not been exposed to it.

Lognormal distributions are most useful where the data range (the difference between the highest and lowest values) of the x-axis is greater than about 10. If the data range is narrow, the lognormal distribution approximates a normal distribution.

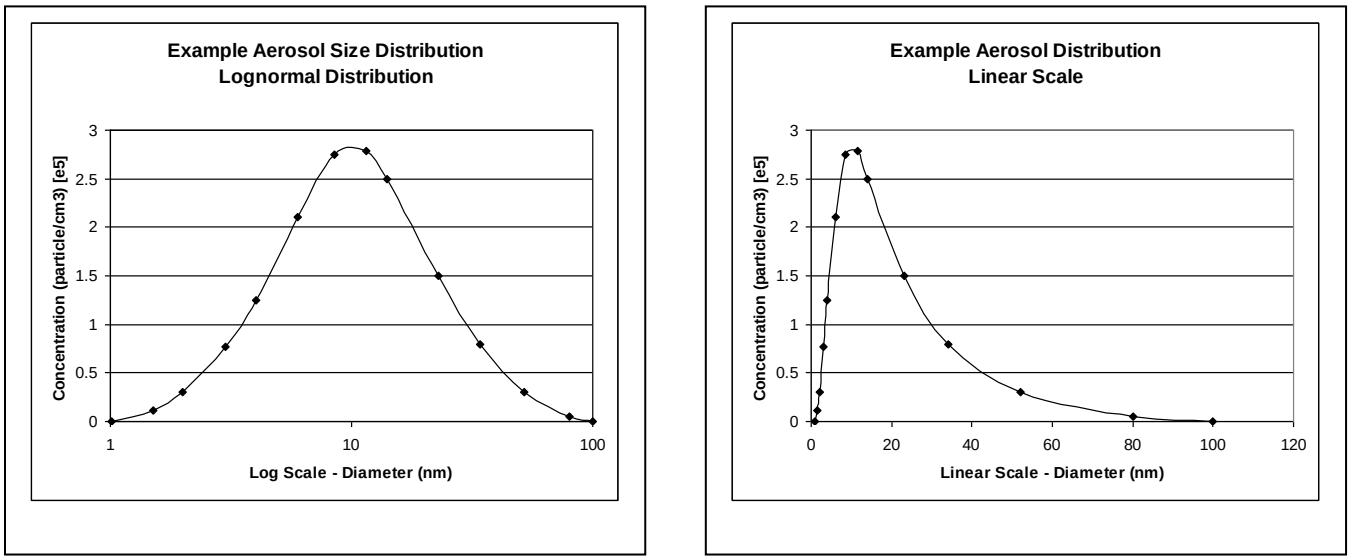

Figure 1: a) Aerosol data displayed using a log scale for particle diameter and b) using a linear scale for particle diameter.

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Lognormal Statistics

Count Median Diameter (*CMD*)

When statistically describing lognormal distributions, the geometric mean diameter (D_g) of normal distributions is replaced by the count median diameter (*CMD*). In lognormal distributions, the log of the particle size distribution is symmetrical, so the mean and the median of the lognormal distribution are equal. The median of the lognormal distribution and normal distribution are equal, since the order of the values does not change when converting to a lognormal distribution. Therefore, for a lognormal distribution, $D_g = CMD$.

$$D_g = CMD = (D_1^{n_1} D_2^{n_2} D_3^{n_2} \cdots D_N^{n_N})^{1/N}$$

where:
D_g = geometric mean diameter
D_i = midpoint particle size
n_i = number of particles in group *i* having a midpoint size D_i
N = $\sum n_i$, the total number of particles, summed over all intervals

Geometric Standard Deviation (σ_g or GSD)

In aerosol lognormal distributions, the log of the particle diameter is normally distributed. The normal, linear based standard deviation (σ) with which most are familiar with is replaced with the standard deviation of the logarithms, called the geometric standard deviation (σ_g or GSD). The geometric standard deviation is always greater than 1 ($\sigma_g \geq 1$).

$$\log \sigma_g = \left[\frac{\Sigma n_i (\log D_i - \log D_g)^2}{N - 1} \right]^{1/2}$$

where:
σ_g = geometric standard deviation (GSD)
D_i = midpoint particle diameter of the i^{th} bin
n_i = number of particles in group *i* having a midpoint size D_i
N = $\sum n_i$, the total number of particles, summed over all intervals

What Do the Lognormal Statistics Mean?

The geometric standard deviation describes how spread out the values are in the distribution. In a normal distribution, 95% of the particle diameters fall within $D_p \pm 2\sigma$. In a lognormal distribution, 95% of the particle diameters fall within a size range expressed as:

$$10^{[\log CMD \pm 2 \log \sigma_g]}$$

or

$$\frac{CMD}{\sigma_g^{\,2}} < 95\% \text{ of all particle diameters} < CMD * \sigma_g^{\,2}$$

i.e., if σ_g = 2.0 and *CMD* = 10nm; 95% of the particle diameters lie between $\left(\frac{10}{4}\right)$ = 2.5nm and $(10 * 4)$ = 40nm. **Monodisperse** aerosols have narrow size distributions and lower geometric standard deviation values (σ_g).
Polydisperse aerosols have wider size distributions and higher geometric standard deviation values (σ_g).

The geometric standard deviation value at which an aerosol can be considered generally monodisperse is quite subjective, but a good rule of thumb is:

Monodisperse:	**$\sigma_g \leq 1.25$**
Polydisperse:	**$\sigma_g > 1.25$**

Many scientists prefer a more stringent definition of monodispersity with $\sigma_g \leq 1.1$. A $\sigma_g \leq 1.05$ is considered highly monodisperse.

Normalized Concentrations: *dN/dlogD$_p$*

Limitations of Concentration (*dN*) vs. Particle Diameter Data Displays

As established above, aerosol distributions are predominantly lognormal in character, so data is typically plotted on a lognormal X-axis. In the simplest technique, particle data is plotted as a function of the concentration (*dN*) for each particle size bin. The mode concentration of the size distribution is often estimated by the concentration in the peak bin. In aerosol sizing instruments, the number of size bins is finite. This simple concentration vs. log particle diameter approach works well only if you are exclusively using one type of instrument, or if you are comparing instruments with identical resolution. However, if you try to compare aerosol size distributions taken of the same aerosol using instruments of different resolutions, plots of concentration vs. particle diameter can be confusing.

In the example below you will see two different plots of data taken with our Scanning Mobility Particle Sizer™ (SMPS™) Spectrometer. On the left the distribution is plotted using 64 channel resolution. On the right is the exact same data plotted with 32-channel resolution. At first glance, it looks like the aerosol represented by the data on the right is roughly double the concentration. However, this is just an illusion from the difference in resolution. The concentration on the right is double because the width of the bin is twice as large as the channel on the left.

The Resolution Problem with Concentration vs. Particle Diameter Data Display

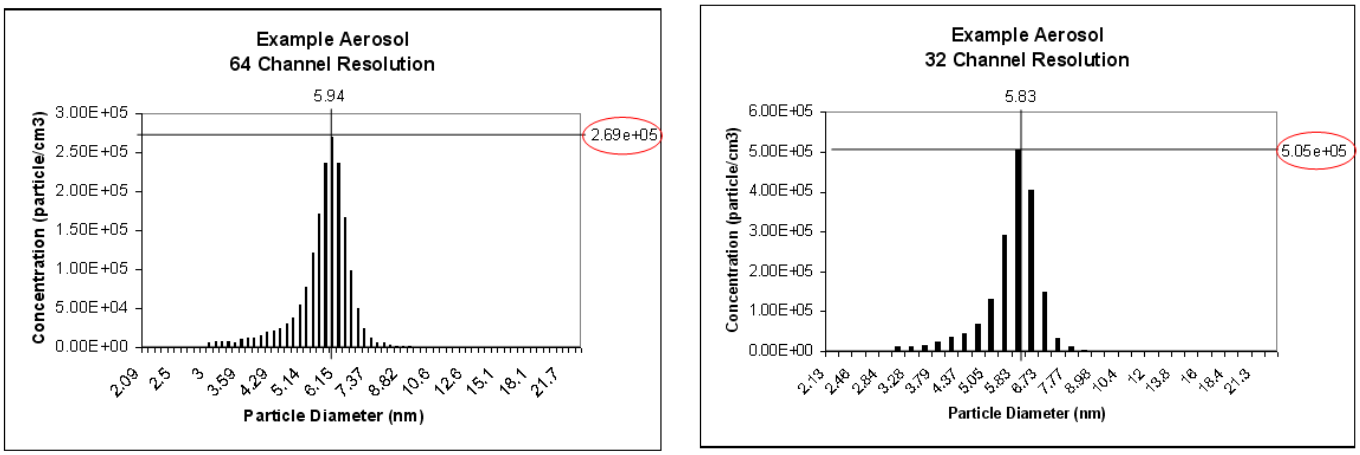


Figure 2: a) SMPS concentration vs. particle diameter data taken with 64 channel resolution and b) 32 channel resolution.

The solution: *dN/dlogD$_p$*

The method statisticians use to avoid this problem is to plot the data using normalized concentration (*dN/dlogD$_p$*). *dN* (or ΔN) is the number of particles in the range (total concentration) and *dlogD$_p$* (or Δ*logD$_p$*) is the difference in the log of the channel width. *dlogD$_p$* is calculated by subtracting the log of the lower bin boundary from the log of the upper boundary for each channel (normalizing for bin width). The concentration is divided by the bin width, giving a normalized concentration value that is independent of the bin width.

$$dN / d\log D_p = \frac{dN}{d\log D_p} = \frac{dN}{\log D_{p,u} - \log D_{p,l}}$$

where:
dN = particle concentration
D_p = midpoint particle diameter
$D_{p,u}$ = upper channel diameter
$D_{p,l}$ = lower channel diameter

With normalized concentrations the concentration values at the mode will be similar even on instruments with very different resolution.

Below is the same SMPS™ spectrometer data plotted using normalized concentration. You can see that the peak concentration values are very similar for both graphs. Note, that normalized concentrations are several orders of magnitude larger than the concentrations not normalized. This is due to the high resolution and small bin width of TSI® particle instruments. The width of the bins (which is the divisor in normalized concentration) is much less than one.

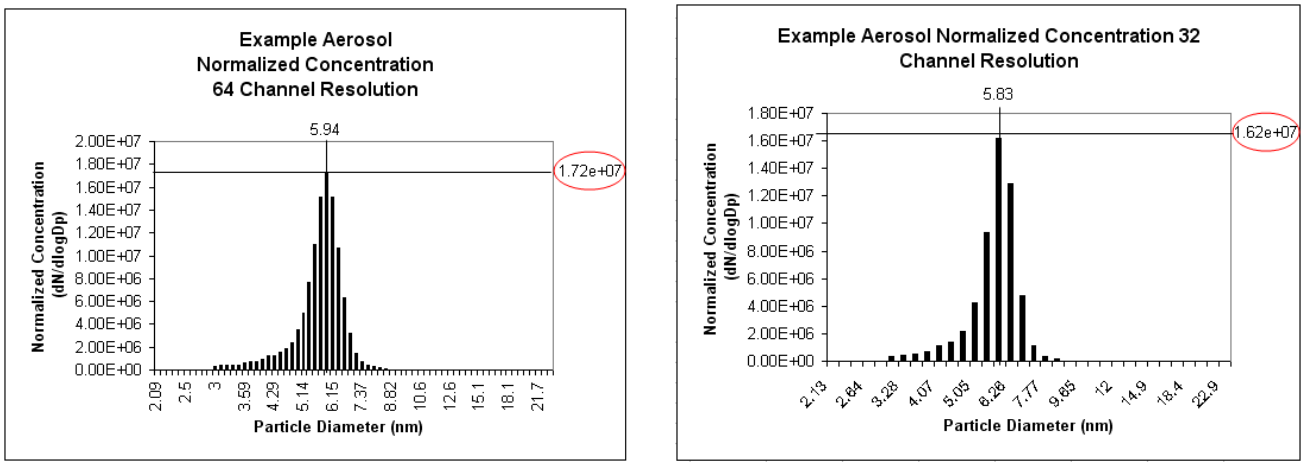


*Figure 3: a) SMPS™ **normalized concentration** vs. particle diameter data taken with 64-channel resolution and b) 32-channel resolution. The small difference in mode concentration is due to slight particle bin changes in the regions near the bin boundaries.*

TSI Instrumentation and *dN/dlogD$_p$*

The simple way to use *dN/dlogD$_p$* with TSI particle sizers is to understand that for most every instrument we have equally spaced size channels (on a log scale). The differences between the boundaries for any one channel are the same as any other channel.

The **Scanning Mobility Particle Sizer™ (SMPS™) Spectrometer** is typically used with 64 channels per decade resolution → *dN/dlogD$_p$* = 1/64. Therefore, to calculate the normalized concentration, the concentration value for each bin is divided by 1/64 (i.e., multiplied by 64). The SMPS™ data can be displayed in 64, 32, 16, 8, or 4 channel resolution.

The **Aerodynamic Particle Sizer®** , the **Ultraviolet Aerodynamic Particle Sizer®**, and the **Single-Box Scanning Mobility Particle Sizer™ (SMPS™) Spectrometer** have 32 channels per decade resolution. Therefore, to calculate the normalized concentration, the concentration value for each bin is divided by 1/32 (i.e., multiplied by 32). The **Engine Exhaust Particle Sizer™** and **Fast Mobility Particle Sizer™** have 16 channels per decade resolution. The displayed resolution in the **Laser Aerosol Spectrometer** is determined by the channel boundaries set by the user.

Description	Model	Resolution	Multiplier
Scanning Mobility Particle Sizer™ (SMPS™) Spectrometer	3936xxx	Adjustable	64, 32, 16, 8, or 4
Aerodynamic Particle Sizer™ (APS™) Spectrometer	3321	Fixed	32
Ultraviolet Aerodynamic Particle Sizer® (UV-APS™) Spectrometer	3314	Fixed	32
Single-Box Scanning Mobility Particle Sizer™ (SMPS™) Spectrometer	3034	Fixed	32
Engine Exhaust Particle Sizer™ (EEPS™) Spectrometer	3090	Fixed	16
Fast Mobility Particle Sizer™ (FMPS™) Spectrometer	3091	Fixed	16
Laser Aerosol Spectrometer	3340	Adjustable	User determined

Common logarithm vs. Natural logarithm

Logarithms can be conceptualized as the inverse of exponents. The logarithm (with defined base) of a number is the power to which the base must be raised in order to produce the number.

Logarithms: $x = b^y \rightarrow y = \log_b(x)$

Common logarithm	$\equiv \log(x) = \log_{10}(x)$
Natural logarithm	$\equiv \ln(x) = \log_e(x)$*

*The irrational, mathematical constant e is ~2.71828...

The natural logarithm (ln) has many convenient computational properties (i.e. derivative = 1/x) which make it attractive to mathematicians. Common log (log) tends to be attractive to engineers since numbers are typically base 10 and mental math is easier.

Fundamentally, logarithms (regardless of the base) are similar to each other. You can easily change from one base to the other.

Recall the following logarithm definitions and properties:

1. $\log(x) = \log_{10}(x)$
2. $\ln(x) = \log_e(x)$
3. $\log_b(x^y) = y \log_b(x)$
4. $x = b^{\log_b(x)}$

So

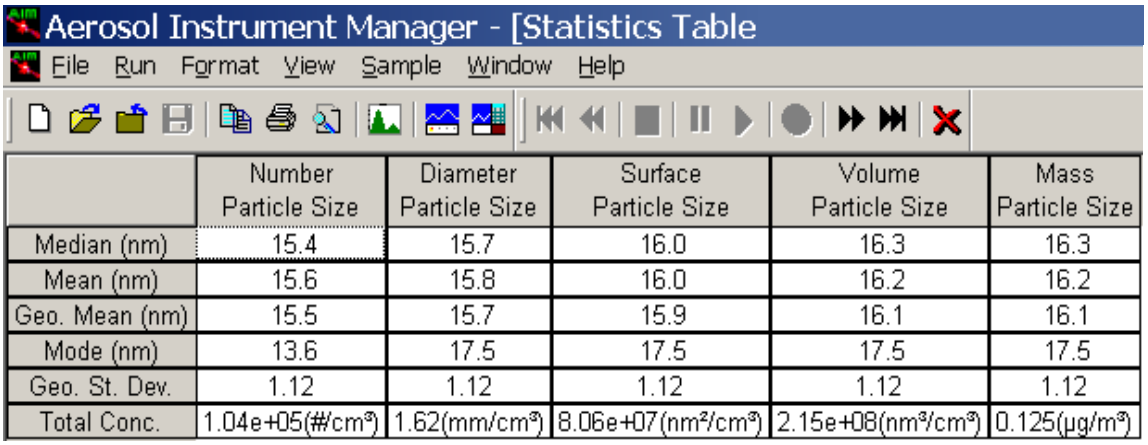
$$\ln(x) = \ln(10^{\log_{10}(x)}) = \log_e(10^{\log_{10}(x)}) = \log_{10}(x)\log_e(10) = \log(x)2.3025$$

$$\ln(x) = 2.3025 \log(x)$$

The only real difference between these two functions is a scale factor (2.3025). Obviously, if you multiply both sides of the equation by the same number the relative values of the constants remains the same on both sides.

In TSI aerosol data analysis, $dlogD_p$ is used as the divisor for normalized concentration values. If $dlnD_p$ was used instead, the absolute values of the normalized concentration numbers would change by a scale factor, but the shape of the distribution would not be affected.

The calculated



	Number Particle Size	Diameter Particle Size	Surface Particle Size	Volume Particle Size	Mass Particle Size
Median (nm)	15.4	15.7	16.0	16.3	16.3
Mean (nm)	15.6	15.8	16.0	16.2	16.2
Geo. Mean (nm)	15.5	15.7	15.9	16.1	16.1
Mode (nm)	13.6	17.5	17.5	17.5	17.5
Geo. St. Dev.	1.12	1.12	1.12	1.12	1.12
Total Conc.	1.04e+05(#/cm³)	1.62(mm/cm³)	8.06e+07(nm²/cm³)	2.15e+08(nm³/cm³)	0.125(µg/m³)

Figure 4: SMPS Aerosol Instrument Manager® (AIM) software statistics box.

Final Note

The primary reference for this application note was the book “Aerosol Technology: Properties, Behavior, and Measurement of Airborne Particles,” by William C. Hinds. This text is a gold standard for aerosol technology fundamentals and is highly recommended to anyone who will be venturing into the field of aerosol science. It is easy to read and understand and covers a thorough working knowledge of modern aerosol theory and applications.

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